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Upper bounds on the torque in cylindrical Couette flow

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Upper bounds on the torque are derived for a fluid that is contained between two concentric rotating cylinders. Absolute upper bounds are obtained by requiring that the fluid satisfy the boundary conditions and the dissipation integral. Improved bounds are then found by requiring that the fluid satisfy continuity conditions. These bounds are in qualitative agreement with the data in that they reflect the asymptotic parameter dependence in the range of experimental data.

1. Introduction

Non-linearities in the Navier–Stokes equations of motion present insurmountable obstacles to any direct approach to statistically steady turbulent flows. Since exact solutions to the Navier–Stokes equations are unattainable for such flows, it is therefore reasonable to seek bounds on the solutions.

Bounding procedures involve the formulation of variational problems, and the stability bounds obtained by Orr (1907), Serrin (1959) and Joseph (1966), for example, illustrate the utility of the variational approach. Upper bounds on the asymptotic behaviour of turbulent flows were first obtained by Howard (1963), who placed an upper bound on the heat transport in turbulent convection. Busse (1968, private communication) applied Howard's method to turbulent shear flow and then extended the method to include solutions consisting of many wave-numbers.

The integral constraints that Howard used consisted of the dissipation integral and the entropy-flux integral. In this paper we use Howard's method to place upper bounds on the torque in cylindrical Couette flow. In a forthcoming paper it will be shown that a more restrictive set of integral constraints leads to improved bounds. Although a sufficiently large number of constraints should provide a very good bound, the greater the number of constraints that are imposed the more difficult the problem becomes. Therefore, the first few terms in a sequence of integral constraints should give a reasonably accurate description of the flow in order for the method to be of value.

The torque that is transmitted to the fluid by the rotation of the concentric cylinders is a measurable quantity, and may be shown to be the only quantity that is independent of the axial direction. It is therefore reasonable to seek an upper bound on the torque. In addition, maximum torque corresponds to maximum dissipation, and it has been suggested by Coles (1965), for example, that the fluid would attempt to reach a state in which the dissipation was a maximum.

2. The equations

We consider an incompressible, homogeneous, and neutrally stratified fluid that is contained between two concentric cylinders of infinite axial extent. The inner cylinder R_i and the outer cylinder R_0 rotate with time-independent angular velocities Ω_i and Ω_0, respectively. The velocity is separated into an average in the circumferential and axial directions and a departure from this average. Perturbation velocity components are indicated by subscripted lower-case letters, where the subscript 1 denotes the circumferential direction, the subscript 2 denotes the axial direction, and the subscript 3 denotes the radial direction. The boundary conditions on the velocity components are

$$\left.\begin{aligned} u_1 = u_2 = u_3 = \frac{\partial u_3}{\partial r} = 0, \\ U = R_i\Omega_i, \end{aligned}\right\} r = R_i,$$

$$\left.\begin{aligned} u_1 = u_2 = u_3 = \frac{\partial u_3}{\partial r} = 0, \\ U = R_0\Omega_0, \end{aligned}\right\} r = R_0.$$

It will be convenient to use a bar to denote an average in the circumferential and axial directions, and a set of brackets to denote an average over the entire volume. The assumption is made that time and space averages may be commuted, so that the time rate of change of a space average is zero.

We shall take as our basic set of equations

$$\left\langle u_1\nabla^2 u_1 + u_2\nabla^2 u_2 + u_3\nabla^2 u_3 - \frac{u_1^2+u_3^2}{r^2} + \frac{2u_1}{r^2}\frac{\partial u_3}{\partial \phi} - \frac{2u_3}{r^2}\frac{\partial u_1}{\partial \phi}\right\rangle$$
$$= T^2\langle\overline{u_1u_3}^2\rangle - \sigma T^2\left\langle\frac{u_1u_3}{r^2}\right\rangle^2 - T\sigma\left\langle\frac{u_1u_3}{r^2}\right\rangle, \quad (1)$$

and

$$N = 1 + T\left\langle\frac{u_1u_3}{r^2}\right\rangle, \quad (2)$$

where the following definitions have been made

$$N = \frac{-(R_0^2-R_1^2)\,G}{4\pi R_i^2R_0^2\nu(\Omega_0-\Omega_i)},$$

$$T = \frac{\Omega_iR_0^2}{\nu}(1-\mu), \quad \mu = \Omega_0/\Omega_i, \quad \eta = R_i/R_0, \quad \sigma = 2\eta^2/(1-\eta^2).$$

Now the first terms on the right side of the energy equation (1) may be written as

$$T^2\langle\overline{u_1u_3}^2\rangle - \sigma T^2\left\langle\frac{u_1u_3}{r^2}\right\rangle^2 \equiv T^2\left\langle\left(\overline{u_1u_3} - \frac{\sigma}{r^2}\left\langle\frac{u_1u_3}{r^2}\right\rangle\right)^2\right\rangle, \quad (3)$$

so that their sum is positive definite. The left side of (1) is negative definite, giving the result that

$$T\left\langle \frac{u_1 u_3}{r^2} \right\rangle \geqslant 0.$$

The quantity N may be interpreted as the equivalent of a Nusselt number; that is, it represents the ratio of the total torque to the laminar torque. Thus, the torque is always greater than or equal to its laminar value. For a given set of external conditions we then have an absolute lower bound on the torque.

3. Asymptotic upper bounds

Upper bounds on the torque were obtained by methods similar to those used by Howard (1963). Modifications of Howard's analysis were necessary because of the difference in geometry and the appearance of additional external parameters. An upper bound on the torque corresponding to Howard's first example in which the continuity condition was not imposed is given by

$$N \leqslant 1 + \frac{3\eta^2(1-\eta^2)}{16}|T|. \tag{4}$$

In the narrow gap limit (4) states that the drag coefficient is a constant, independent of the Reynolds number. The physical picture to which this would correspond is a flow over a rough boundary.

The upper bound given by (4), while not a particularly good approximation to the torques that have been observed experimentally, does constitute a formal upper bound for all flows that satisfy the stated conditions. If one imposes the additional condition that $\nabla . \mathbf{v} = 0$ and considers the case in which all perturbation velocities have a single wave-number periodicity in the axial direction, one obtains the result

$$N < \left|\frac{\Omega_1 R_1^2(1-\mu)}{\nu}\right|^{\frac{3}{4}} \frac{1-\eta^2}{2(0{\cdot}337)^{\frac{3}{2}} 8^{\frac{7}{4}} (\ln(1/\eta))^{\frac{1}{4}} (1+\eta^{\frac{7}{3}})^{\frac{3}{2}}}. \tag{5}$$

In order to compare the asymptotic relation (5) with the experimental data, it will be convenient to define a Reynolds number

$$R = \frac{\Omega_i R_i (R_0 - R_i)(1-\mu)}{\nu}.$$

Equation (5) may then be written (for $\mu = 0$)

$$N = R^{\frac{3}{4}} F(\eta), \tag{6}$$

where

$$F(\eta) = \frac{\eta^{\frac{3}{4}}(1+\eta)(1-\eta)^{\frac{1}{4}}}{2(0{\cdot}337)^{\frac{3}{2}} 8^{\frac{7}{4}} (\ln(1/\eta))^{\frac{1}{4}} (1+\eta^{\frac{7}{3}})^{\frac{3}{2}}}. \tag{7}$$

Surprisingly enough, the power laws for cylindrical Couette flow, when properly interpreted, turn out to be identical to those found by Howard. Moreover, the appearance of additional parameters, namely μ and η, gives the problem of cylindrical Couette flow a richer structure than the problem Howard studied.

The $\frac{3}{8}$ power law derived by Howard becomes a $\frac{3}{4}$ power law in this problem (since the Rayleigh number is replaced by the square of a Reynolds number). In the narrow gap limit, (6) corresponds to a drag coefficient that is proportional to $R^{-\frac{1}{4}}$. This power law is characteristic of the Blasius régime associated with flow in a channel. Figure 1 contains a plot of $F(\eta)$ that attains a maximum value at approximately $\eta = 0{\cdot}81$. The asymptotic value of N for $\eta = \frac{1}{2}$, for example, is less than that for $\eta = 0{\cdot}85$. This asymptotic behaviour also occurs in the data for R greater than R_c. Even though the critical Reynolds number for $\eta = \frac{1}{2}$ is less than that for $\eta = 0{\cdot}85$, Donnelly's (1958) torque data for $\eta = \frac{1}{2}$ cuts through

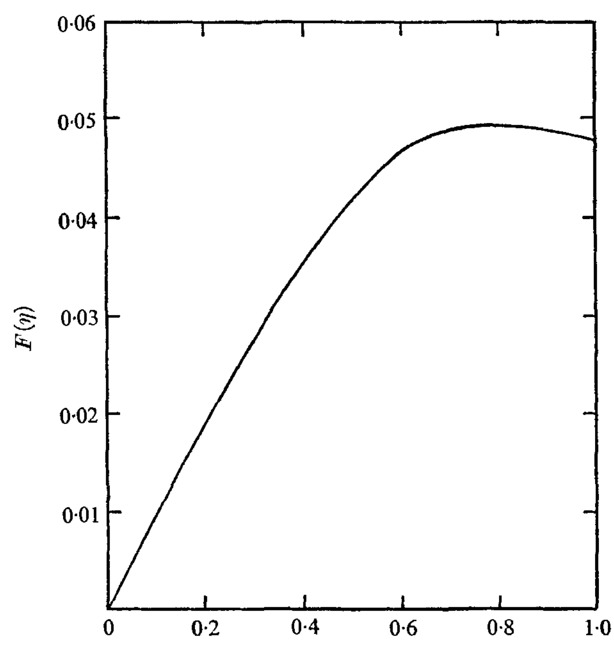

FIGURE 1. Asymptotic constant of proportionality.

the data of Wendt (1933) for $\eta = 0{\cdot}68$ and $0{\cdot}85$. Furthermore, the increase in torque with Reynolds number for $\eta = \frac{1}{2}$ is considerably smaller than that observed for larger values of η.

The question of whether or not (6) is also an upper bound for a multi-alpha solution has been answered by Busse (private communication). Busse's investigation was oriented toward the study of Poiseuille flow and plane Couette flow. Nevertheless, his methodology is directly applicable to the problem considered in this paper.

The boundary layer for the single alpha solution may be thought of as having a boundary layer that is scaled by a second wave-number. This boundary layer in turn has a boundary layer that is scaled by a third wave-number, *ad infinitum*. Busse found that the ratio of the length scales for the first and second modes was of the order 30. For a larger number of modes the ratio between successive scales decreases so that, for example, the length ratio between the sixth and seventh modes is approximately 2. By applying Busse's procedure to the problem at hand, one finds that for values of R much beyond 10^4, it will be necessary to include more than a single wave-number.

4. Critical and near critical torques

The Euler–Lagrange equations obtained by maximizing the torque subject to the dissipation integral (1) and the continuity equation are as follows:

$$\nabla^2 u_1 - \frac{u_1}{r^2} = -\frac{Tu_3}{2}\left\{\frac{\sigma N}{r^2} + \frac{T\langle \overline{u_1 u_3^2}\rangle}{r^2\langle u_1 u_3/r^2\rangle} - 2T\overline{u_1 u_3} + \frac{1}{r}\frac{\partial \Gamma}{\partial \phi}\right\}, \tag{8}$$

$$\nabla^2 u_3 - \frac{u_3}{r^2} = -\frac{Tu_1}{2}\left\{\frac{\sigma N}{r^2} + \frac{T\langle \overline{u_1 u_3^2}\rangle}{r^2\langle u_1 u_3/r^2\rangle} - 2T\overline{u_1 u_3} + \frac{\partial \Gamma}{\partial r}\right\}, \tag{9}$$

$$\nabla^2 u_2 = \frac{\partial \Gamma}{\partial z}, \tag{10}$$

where Γ is an unspecified Lagrange multiplier. It is important to notice that (8) to (10) are separable. We then consider a simple separation in the axial direction in which all disturbances are characterized by a single wave-number.

The optimal equations may then be written as

$$(DD_* - \alpha^2)u_1 = -\frac{Tu_3}{2}\left\{\frac{\sigma N}{r^2} + \frac{T\langle \overline{u_1 u_3^2}\rangle}{r^2\langle u_1 u_3/r^2\rangle} - 2T\overline{u_1 u_3}\right\}, \tag{11}$$

$$(DD_* - \alpha^2)^2 u_3 = \frac{\alpha^2 T u_1}{2}\left\{\frac{\sigma N}{r^2} + \frac{T\langle \overline{u_1 u_3^2}\rangle}{r_2\langle u_1 u_3/r_2\rangle} - 2T\overline{u_1 u_3}\right\}, \tag{12}$$

where $D = \dfrac{d}{dr}$ and $D_* = \dfrac{1}{r}\dfrac{d}{dr}$.

At the critical point, (11) and (12) reduce to

$$(DD_* - \alpha^2)u_1 = -\frac{Tu_3}{2}\frac{\sigma}{r^2}, \tag{13}$$

$$(DD_* - \alpha^2)u_3 = \frac{\alpha T u_1 \sigma}{2r^2}. \tag{14}$$

The critical value of T^2 is given by

$$T_c^2 = \min \frac{4\langle -u_1(DD_* - \alpha^2)u_1\rangle\langle u_3(DD_* - \alpha^2)^2 u_3\rangle}{\sigma^2\alpha^2\langle u_1 u_3/r^2\rangle^2}. \tag{15}$$

Critical values of T^2 for the specific cases $\eta = 0{\cdot}5$ and $0{\cdot}75$ have been obtained by expanding the velocity components in a series of Bessel functions, and then integrating over the volume to obtain a set of algebraic equations that are linear in the expansion coefficients. Standard eigenvalue methods were then used to find the critical value of T^2 for different values of the wave-number, α, thereby leading to a minimum value for T^2.

Equations (11) and (12) were used to obtain the initial finite amplitude dependence of N on Reynolds number. The procedure for obtaining N was similar to that used in the stability problem.

Numerical calculations of the torque for super critical Reynolds numbers were not carried out beyond $N = 1{\cdot}45$. For N greater than $1{\cdot}45$ the numerical iteration scheme would not converge. In an effort to bridge the gap between the numerical calculations and the asymptotic predictions at high Reynolds numbers, the numerical calculations were extended by using the shape assumption.

Using a method first suggested by Malkus & Veronis (1958), Stuart (1958) and later Davey (1962) determined a relationship between torque and Taylor number for super critical values of the Taylor number. They assumed that to a first approximation the perturbation velocity components for super critical Taylor numbers may be given by a function whose radial dependence is that of the critical eigenfunction.

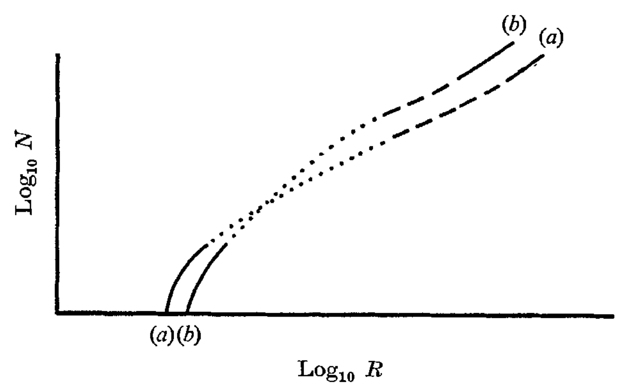

FIGURE 2. Upper bounds on the torque. (*a*) $\eta = 0{\cdot}50$; (*b*) $\eta = 0{\cdot}75$.

The predicted curves were in good agreement with the data near the critical Taylor number, but departed from the data for Taylor numbers that exceeded critical by about 10%. The aggreement between theory and experiment over such a range of Taylor numbers is now thought to be fortuitous, and the shape assumption is not regarded very highly at the present time.

The torque curves for $\eta = 0{\cdot}5$ and $0{\cdot}75$ are shown in figure 2. The lower solid lines represent the numerical calculations. The dotted segments represent the results of the shape assumption calculations. The upper solid lines represent the asymptotic predictions, and the dashed sections represent a smoothed transition between the shape assumption curve and the asymptotic predictions.

5. Comparison with previous data

For values of N greater than unity, plots of N *vs.* R for different values of η reveal an interesting phenomenon (see figure 3). Donnelly's curve for $\eta = 0{\cdot}5$ has a much smaller slope than the other curves, and cuts across them. In fact, the strong dissimilarity between Donnelly's data and that of Wendt and Taylor might lead one to discard the results of Donnelly's experiment.

There are several reasons, however, for believing that this strange result is characteristic of large gap spacings. The reasons for believing this are fourfold: (i) Wendt's data for $\eta = 0{\cdot}68$ cut across his own data for $\eta = 0{\cdot}85$; (ii) Donnelly's data is in agreement with Davey's (1962) finite amplitude predictions; (iii) the asymptotic value of N from (6) is significantly lower at $\eta = \frac{1}{2}$ than at $\eta = 0{\cdot}75$, say; (iv) the numerical calculations for $\eta = \frac{1}{2}$ and $\eta = 0{\cdot}75$ indicate a similar crossing-over.

Taylor's (1936) data have been included in order to provide a check on the accuracy of Wendt's data (i.e. Taylor's curve for $\eta = 0{\cdot}852$) and also to indicate the position of the curves for smaller gaps. The slopes of Taylor's curves for log N close to zero should be considered only as rough approximation, however, due to the small number of data points that were available.

An absolute upper bound on N is given by (4). This result has been plotted in figure 4 for several values of η. Equation (4) results in torques that are one or two orders of magnitude larger than those determined experimentally.

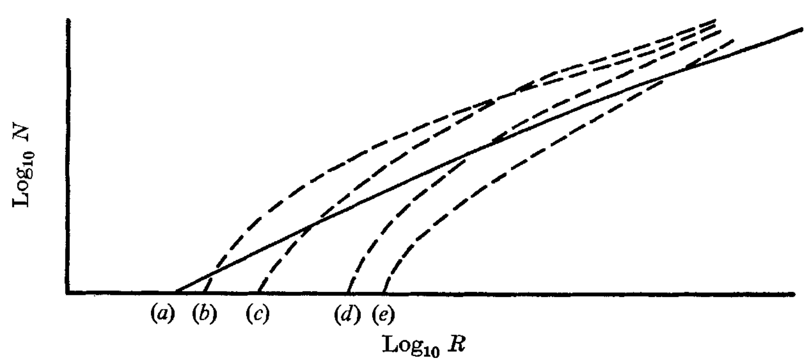

FIGURE 3. Super critical torque curves. (*a*) Donnelly (1958), 0·5; (*b*) Wendt (1933), 0·68; (*c*) Wendt (1933), 0·85; (*d*) Wendt (1933), 0·94; (*e*) Taylor (1936), 0·965.

An upper bound on the torque for a disturbance characterized by a single wavenumber is given by (6). This result has also been incorporated into figure 4, and is seen to provide a significant improvement in that the torques now exceed the experimental values by at most a factor of three or four.

In addition to providing an improved upper bound, the asymptotic predictions reflect the reversal of the $N(R)$ curves with η. The asymptotic curve for $\eta = 0{\cdot}5$ lies under the $\eta = 0{\cdot}68$ curve by approximately the same amount as for the experimental curves. Furthermore, the asymptotic slopes are similar to the experimental slopes in the range where the theory is valid. As was mentioned in §4, multi-alpha solutions will have to be considered for Reynolds numbers greater than about 10^4. This phenomenon is apparently reflected in Taylor's narrow gap curves (i.e. $\eta = 0{\cdot}965$ and $\eta = 0{\cdot}973$) where the slope approaches that given by (4).

In the limit of very large Reynolds numbers, N apparently becomes directly proportional to the Reynolds number, in agreement with Busse's result that in the asymptotic limit the drag coefficient becomes inversely proportional to log R.

It will be shown in a forthcoming paper that for the case $\mu = 0$, the critical value of T^2 (or equivalently R^2) obtained from the total dissipation integral should agree with that found from the separate components of the dissipation integral at a value of η approximately equal to 0·5. The results of $R = 67{\cdot}2$ and $\alpha = 6{\cdot}3$ when the values of $\mu = 0$ and $\eta = 0{\cdot}5$ are inserted into (13) and (14) are within a few percent of the values quoted by Chandrasekhar (1958) and others. However, for values of μ different from zero the discrepancies become large, particularly for large negative values of μ. For $\mu = 0$ and $\eta = 0{\cdot}75$ the computed Reynolds number is 73·2, compared with the value of 85·7 from Sparrow, Munro & Jonsson (1964). As η approaches unity, the actual critical

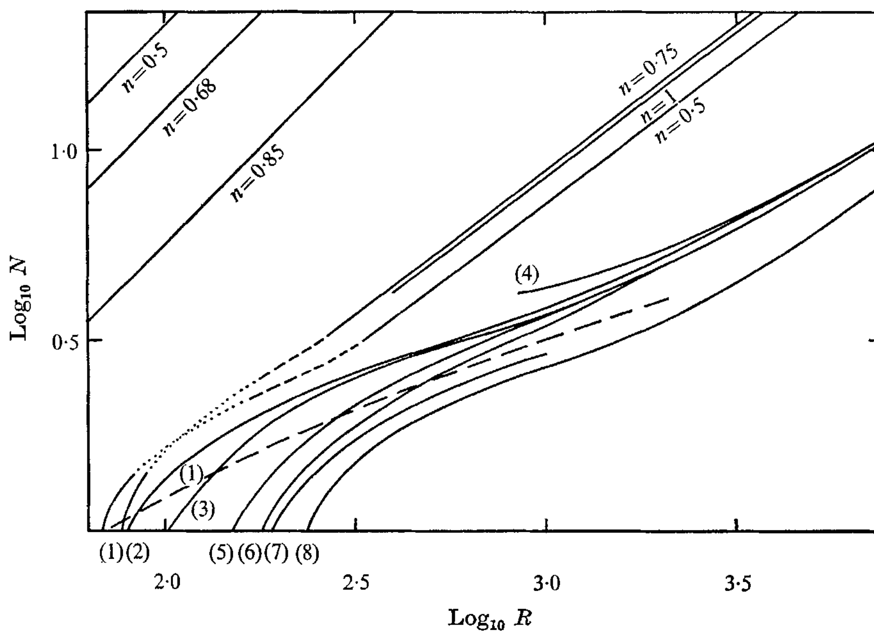

FIGURE 4. Experimental torque data and predicted upper bounds. (1) Donnelly (1958), $\eta = 0{\cdot}5$, (2) Wendt (1933), $\eta = 0{\cdot}68$; (3) Wendt (1933), $\eta = 0{\cdot}85$; (4) Taylor (1936), $\eta = 0{\cdot}852$; (5) Wendt (1933), $\eta = 0{\cdot}935$; (6) Taylor (1936), $\eta = 0{\cdot}960$; (7) Donnelly (1958), $\eta = 0{\cdot}95$; (8) Taylor (1936), $\eta = 0{\cdot}973$.

values continue to increase, but the values from (13) and (14) approach the limiting value $2\sqrt{1708}$. (In the narrow gap limit, (13) and (14) become identical in form to the equations for the Rayleigh problem with $(R/2)^2$ assuming the role of a Rayleigh number.)

6. Concluding remarks

An upper bound on the torque has been derived subject to the conditions that the fluid satisfy the boundary conditions and the dissipation integral. Reductions in the bound were obtained by imposing the continuity equation.

The failure of the Euler–Lagrange equations (11) and (12) to adequately represent the Taylor mechanism at the critical point is due to the particular integral constraints that were used. The Taylor mechanism is an important energy source for individual fluid elements but cannot change the energy of the fluid as a whole. This failure can be overcome, however, by using the separate dissipation integrals as constraints. The use of additional integral constraints will be discussed in a forthcoming paper.

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